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ABSTRACT. We study the moments of the symmetric power L-functions of primitive forms at the edge of the critical strip twisted by the square of the value of the standard L-function at the center of the critical strip. We give a precise expansion of the moments as the order goes to infinity.

CONTENTS

1. INTRODUCTION

Let ρ be a representation on $\mathrm{SU}(2)$. For any $g \in \mathrm{SU}(2)$ we define the polynomial

$$D(X,\rho,g) = \det\left(I - X\rho(g)\right)^{-1}. \tag{1}$$

Endowing $\mathrm{SU}(2)$ with its Haar measure, Cogdell & Michel remarked that

$$\int_{\mathrm{SU}(2)} D(X,\rho,g)^z \,\mathrm{d}g = 1 + \left[\frac{z^2}{2}\mathrm{FrSc}(\rho)^2 + \frac{z}{2}\mathrm{FrSc}(\rho)\right] X^2 + O_z\left(X^3\right)$$

[CM04, (2.26)] for any complex number z, where $\mathrm{FrSc}\,\rho$ is the Frobenius-Schur indicator of ρ. The coefficient of X^2 is then

$$\begin{cases} 0 & \text{if } \rho \text{ is not self-dual,} \\ \dfrac{z(z-1)}{2} & \text{if id appears once in the irreducible decomposition of } \mathrm{Sym}^2\rho, \\ \dfrac{z(z+1)}{2} & \text{if id appears once in the irreducible decomposition of } \wedge^2\rho. \end{cases}$$

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For $\rho = \mathrm{St}$ (the standard representation of SU(2)), this coefficient is $\frac{z(z-1)}{2}$. In particular, the Euler product (indexed over the set $\mathscr{P}$ of all prime numbers)

$$\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)} D\left(p^{-1/2},\mathrm{St},g\right)^z\,\mathrm{d}g$$

converges only for $z\in\{0,1\}$.

Let $k\geq 2$ be a (fixed) even integer. For any squarefree integer N such that the set of primitive forms of weight k over $\Gamma_0(N)$ is not empty, we denote by $\mathrm{H}_k^*(N)$ this set. To any $f\in\mathrm{H}_k^*(N)$ we associate an L-function defined by the Euler product

$$L(s,f)=\prod_{p\in\mathscr{P}}\det\left(I-X\,\mathrm{St}(g_f(p))p^{-s}\right)^{-1}$$

where for any prime number p, the matrix

$$g_f(p)=\begin{pmatrix}\alpha_f(p) & 0\\ 0 & \beta_f(p)\end{pmatrix}$$

is made up of the local parameters in p associated to f. For any prime p not dividing N, this matrix belongs to SU(2) and for the $\omega(N)$ prime numbers dividing N we have $\alpha_f(p)=\pm p^{-1/2}$ and $\beta_f(p)=0$. Hence it tempts naturally to model the moments of L-functions for the primitive forms in $\mathrm{H}_k^*(N)$ (over the discrete harmonic measure) with Euler product of polynomial of type (1) with g in SU(2) endowed with its Haar measure.

As in [CM04], denote by $\sum^{\mathrm{h}}$ the harmonic average. It is apparent that

$$\lim_{N\to+\infty}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^0=\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)} D\left(p^{-1/2},\mathrm{St},g\right)^0\,\mathrm{d}g$$

and it follows from [RW07, Theorem A and Proposition B] that

$$\lim_{N\to+\infty}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^1=\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)} D\left(p^{-1/2},\mathrm{St},g\right)^1\,\mathrm{d}g.$$

The generalization to high power moments sounds problematic, and in fact, there is a convergence problem on the right side. For $z=2$, the lack of convergence of the product in the representation side comes from the term $1/p$ so a natural remedy is natural to consider the normalized form

$$\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)}\left(1-\frac{1}{p}\right)D\left(p^{-1/2},\mathrm{St},g\right)^2\,\mathrm{d}g.$$

To fix ideas, we assume temporarily N to be prime. It turns out that the remedy is appropriate; in fact,

$$\begin{aligned}\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^2 &\sim\left[\prod_{p\in\mathscr{P}}\left(1-\frac{1}{p}\right)\int_{\mathrm{SU}(2)} D\left(p^{-1/2},\mathrm{St},g\right)^2\mathrm{d}g\right]\log N\quad(N\to+\infty)\\ &\sim \mathrm{e}^{-\gamma}\prod_{p\leqslant N}\int_{\mathrm{SU}(2)} D\left(p^{-1/2},\mathrm{St},g\right)^2\mathrm{d}g\quad(N\to+\infty),\end{aligned}$$

where γ is the Euler constant. In other words, we may model $L(1/2,f)^2$ by the product over prime numbers $p\leqslant N$ of the random variables $g\mapsto D\left(p^{-1/2},\mathrm{St},g\right)^2$ with a correction factor $\mathrm{e}^{-\gamma}$.

Our result is actually more precise and we compute all the complex moments of $L(1,\mathrm{Sym}^m f)$ twisted by $L(1/2,f)^2$ without too heavy restriction on the level N. To give our results, we need a few notation.

For any integer $m\geqslant 1$, the mth symmetric power L-function of $f\in \mathrm{H}_k^*(N)$ is

$$L\big(s,\mathrm{Sym}^m f\big)=\prod_{p\in\mathscr{P}}\det\big(I-\mathrm{Sym}^m\rho\left(g_f(p)\right)p^{-s}\big)^{-1}.$$

If $m\in\{1,2,4\}$ it is known to have all the required properties to be an L-function in the sense of [IK04, §5.1] and to have no Landau-Siegel zero [GJ78, Kim03, KS02]. For other values of m, we impose two standard hypothesis - Hypothesis $\mathrm{Sym}^k f$ and $\mathrm{LSZ}^k f$ in [CM04]. Therefore, our results are unconditional for $m\in\{1,2,4\}$ and rest on the standard conjectures for all other cases. We write, γ_∞ for the gamma factor of $L(s,f)$ which depends only on the weight of f. Explicitly it is given by

$$\gamma_\infty(s)=\pi^{-s}\Gamma\left(\frac{s}{2}+\frac{k-1}{4}\right)\Gamma\left(\frac{s}{2}+\frac{k+1}{4}\right).$$

Let

$$F^z(w,s;X)=\big(1-X^{1+2w}\big)\int_{\mathrm{SU}(2)}D\big(X^{1/2+w},\mathrm{St},g\big)^2 D\big(X^{1+s},\mathrm{Sym}^m,g\big)^z\,\mathrm{d}g$$

and

$$\begin{aligned}&C^z(w,s;X)\\&=\begin{cases}\big(1+X^{2+2w}\big)\big(1-X^{2+2w}\big)^{-2}\big(1-X^{1+m/2+s}\big)^{-z} & \text{if } 2\mid m,\\ \dfrac{\big(1+X^{1+w}\big)^{-2}\big(1-X^{1+m/2+s}\big)^{-z}+\big(1-X^{1+w}\big)^{-2}\big(1+X^{1+m/2+s}\big)^{-z}}{2} & \text{if } 2\nmid m.\end{cases}\end{aligned}$$

The function $C^z(w,s;X)$ will be used as a correction factor to $F^z(w,s;X)$. Moreover we define

$$A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)=\prod_{\substack{p\in\mathscr{P}\\ p\nmid N}}F^z\left(0,0;\frac{1}{p}\right)\prod_{\substack{p\in\mathscr{P}\\ p\mid N}}C^z\left(0,0;\frac{1}{p}\right)$$

and

$$B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)=\frac{\mathrm{d}}{\mathrm{d}w_{\,|w=0}}\left(\prod_{\substack{p\in\mathscr{P}\\ p\nmid N}}F^z\left(w,0;\frac{1}{p}\right)\prod_{\substack{p\in\mathscr{P}\\ p\mid N}}C^z\left(w,0;\frac{1}{p}\right)\right).$$

Finally denote by $\varphi(n)$ (resp. $\mu(n)$) the Euler function (resp. Möbius) and by $\log_k$ the k-fold iterated logarithm.

Below are our main results.

***Theorem* A**– *Let $m \in \{1,2,4\}$. There exists two positive real numbers c_m and δ_m such that for any sufficiently large squarefree N,*

$$\begin{aligned}\frac{N}{\varphi(N)}\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac12,f\right)^2 L\left(1,\mathrm{Sym}^m f\right)^z &= A^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m;N\right)\log N\\ &\quad + 2A^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m;N\right)\left(\gamma+\frac{\gamma_\infty'}{\gamma_\infty}\left(\frac12\right)+\sum_{p\mid N}\frac{\log p}{p-1}\right)\\ &\quad + B^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m;N\right)+O_m\left(\exp\left(-\delta_m\frac{\log N}{\log_2 N}\right)\right)\end{aligned}$$

uniformly in

$$|z|\leqslant c_m\frac{\log N}{\log_2 N\log_3 N}.$$

This theorem is proved in Section 3.1. The dependance on the level can be easily depicted when N has no small prime factors. Consider the set of numbers

$$\mathscr{N}(h)=\left\{N\in\mathbb{Z}_{>0} : \mu(N)^2=1 \text{ and } P^-(N)\geqslant h(N)\right\}$$

for some function h where $P^-(N)$ is the smallest prime factor of N with the convention $P^-(1)=+\infty$. We write

$$\begin{aligned}A^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m\right) &= A^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m;1\right)\\ &= \prod_{p\in\mathscr{P}}\left(1-\frac1p\right)\int_{\mathrm{SU}(2)} D\left(\frac{1}{p^{1/2}},\mathrm{St},g\right)^2 D\left(\frac1p,\mathrm{Sym}^m,g\right)^z \mathrm{d}g\end{aligned}$$

and

$$\begin{aligned}B^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m\right) &= B^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m,1\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0}\prod_{p\in\mathscr{P}}\left(1-\frac{1}{p^{1+2w}}\right)\int_{\mathrm{SU}(2)} D\left(\frac{1}{p^{1/2+w}},\mathrm{St},g\right)^2 D\left(\frac1p,\mathrm{Sym}^m,g\right)^z \mathrm{d}g.\end{aligned}$$

***Corollary* B**– *Let $m\in\{1,2,4\}$. There exists a positive real number c_m such that for any sufficiently large squarefree $N\in\mathscr{N}\left(\log^2\right)$,*

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac12,f\right)^2 L(1,\mathrm{Sym}^m f)^z=(1+o_m(1))A^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m\right)\log N$$

uniformly in

$$|z|\leqslant c_m\frac{\log N}{\log_2 N\log_3 N}.$$

This is shown in Section 3.2.

It is interesting to evaluate the asymptotic behavior of the main term

$$A^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m\right)$$

and the constant term $B^{2,z}\left(\frac12,1;\mathrm{St},\mathrm{Sym}^m\right)$ as the exponent $z\to+\infty$ in real numbers.

Let X_m be the Chebyshev polynomial of second kind whose restriction on $[-2,2]$ is defined by

$$X_m(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

They come up naturally in theory of modular forms since, if $\{\chi_{\mathrm{Sym}^m} : m \in \mathbb{Z}_{\geqslant 0}\}$ is the set of irreducible characters of $\mathrm{SU}(2)$, then

$$\chi_{\mathrm{Sym}^m}(g) = X_m(\mathrm{tr}\, g).$$

Let us introduce some auxiliary functions.

$$g_m(t) := \log \int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\mathrm{tr}\, g)}\,\mathrm{d}g = \log\left(\frac{2}{\pi}\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} \sin^2\theta\,\mathrm{d}\theta\right) \quad (t \geqslant 0), \tag{2}$$

$$\widetilde{g}_m(t) := \begin{cases} g_m(t) & \text{if } 0 \leqslant t < 1, \\ g_m(t) - (m+1)t & \text{if } t \geqslant 1, \end{cases} \tag{3}$$

and

$$h_m(t) := \frac{\displaystyle\int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\mathrm{tr}\, g)}\,\mathrm{tr}\, g\,\mathrm{d}g}{\displaystyle 2\int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\mathrm{tr}\, g)}\,\mathrm{d}g} = \frac{\displaystyle\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} \cos\theta \sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} \sin^2\theta\,\mathrm{d}\theta} \quad (t \geqslant 0), \tag{4}$$

$$\widetilde{h}_m(t) := \begin{cases} h_m(t) & \text{if } 0 \leqslant t < 1, \\ h_m(t) - 1 & \text{if } t \geqslant 1. \end{cases} \tag{5}$$

***Theorem* C**– *Let $J \geqslant 1$ and $m \geqslant 1$ be two fixed integers. Then we have*

$$\log A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) = z\left\{(m+1)\log_2 z + (m+1)\gamma + \sum_{j=1}^{J} \frac{a_j}{(\log z)^j} + O\left(\frac{1}{(\log z)^{J+1}}\right)\right\}$$

uniformly for $z \geqslant 3$, where γ is the Euler constant and

$$a_j := \int_0^{+\infty} \frac{\widetilde{g}_m(t)}{t^2}(\log t)^{j-1}\,\mathrm{d}t.$$

The implied constant depends on J and m only.

Theorem C is proved in Section 4.1.

***Theorem* D**– *We have*

$$B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) \ll A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\log z$$

uniformly for $z \geqslant 3$ if m is even; and

$$B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right) = A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\left\{b_m + O\left(\mathrm{e}^{-\sqrt{\log z}}\right)\right\}\sqrt{z}$$

uniformly for $z \geqslant 3$ if m is odd, where

$$b_m := -4\left(2 + \int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t\right) \neq 0.$$

The implied constants depend on m only.

Section 4.2 is devoted to its proof.

It is surprising that the asymptotic behavior of $\log B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m\right)$ changes dramatically according as the parity of m.

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2. PRELIMINARY RESULTS

For every $g\in\mathrm{SU}(2)$, define $\lambda^{z,\nu}_{\mathrm{Sym}^m}(g)$ by the expansion

$$D(X,\mathrm{Sym}^m,g)^z=\sum_{\nu=0}^{+\infty}\lambda^{z,\nu}_{\mathrm{Sym}^m}(g)X^\nu.$$

We have from [RW07, (46) and (36)],

$$\lambda^{z,\nu}_{\mathrm{Sym}^m}(g)=\sum_{u=0}^{m\nu}\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}\chi_{\mathrm{Sym}^u}(g)$$

with

$$\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}=\int_{\mathrm{SU}(2)}\lambda^{z,\nu}_{\mathrm{Sym}^m}(g)\chi_{\mathrm{Sym}^u}(g)\,\mathrm{d}g. \tag{6}$$

One should remark $\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}=0$ for $n>m\nu$. Recall that $\{\chi_{\mathrm{Sym}^m}:\ m\in\mathbb{Z}_{\geqslant 0}\}$ is explicitly defined by the generating series

$$\sum_{m\geqslant 0}\chi_{\mathrm{Sym}^m}(g)T^m=\frac{1}{(1-\alpha T)(1-\overline{\alpha}T)}=D(T,\mathrm{St},g) \tag{7}$$

where α and $\overline{\alpha}$ are the eigenvalues of g. It follows from the study of Cogdell & Michel [CM04] (see also [RW07, eq. (38), (39) and (52)]) that

$$\mu^{z,0}_{\mathrm{Sym}^m,\mathrm{Sym}^u}=\delta(u,0), \tag{8}$$

$$\mu^{z,1}_{\mathrm{Sym}^m,\mathrm{Sym}^u}=z\delta(u,m), \tag{9}$$

$$|\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|\leqslant\binom{(m+1)|z|+\nu-1}{\nu}. \tag{10}$$

2.1. **Combinatorial results.** The aim of this short section is to prove the two following useful equalities:

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1+w)u}}\sum_{\substack{\nu\geqslant 0\\ u\equiv m\nu\,(\mathrm{mod}\ 2)}}\frac{\tau_z(p^\nu)}{p^{(1+m/2+s)\nu}}=C^z\left(w,s;\frac{1}{p}\right), \tag{11}$$

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1/2+w)u}}\sum_{\nu\geqslant 0}\frac{\mu^{z,\nu}_{\mathrm{Sym}^m,\mathrm{Sym}^u}}{p^{(1+s)\nu}}=F^z\left(w,s;\frac{1}{p}\right). \tag{12}$$

Thanks to (10) and the binomial theorem, the series in (12) is absolutely convergent for $\Re s>-1/2$ and $\Re w>-1/2$.

Equality (11) follows directly from the following expressions:

$$\sum_{\substack{u\geqslant 0\\ u \text{ odd}}}(u+1)X^u=\frac{2X}{(1-X^2)^2},\qquad \sum_{\substack{u\geqslant 0\\ u \text{ even}}}(u+1)X^u=\frac{1+X^2}{(1-X^2)^2}$$

and

$$\sum_{\substack{v\geqslant 0\\ v\equiv r\,(\mathrm{mod}\ 2)}}\binom{v+z-1}{v}X^v=\frac{(1-X)^{-z}+(-1)^r(1+X)^{-z}}{2}$$

for any $r\in\{0,1\}$.

From (6) we deduce

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{(1/2+w)u}}\sum_{v\geqslant 0}\frac{\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}}{p^{(1+s)v}}=\int_{\mathrm{SU}(2)}D\left(\frac{1}{p^{1+s}},\mathrm{Sym}^m,g\right)^z\sum_{u\geqslant 1}\frac{(u+1)\chi_{\mathrm{Sym}^u}(g)}{p^{(1/2+w)u}}\,\mathrm{d}g.$$

Let $g\in\mathrm{SU}(2)$ and let $\alpha,\overline{\alpha}$ be its eigenvalues. We use (7) to get

$$\sum_{u\geqslant 1}(u+1)\chi_{\mathrm{Sym}^u}(g)T^u=\frac{\mathrm{d}}{\mathrm{d}T}\frac{T}{(1-\alpha T)(1-\overline{\alpha}T)}=(1-T^2)D(T,\mathrm{St},g)^2.$$

This gives (12).

2.2. **Analytical results.**

***Lemma* 2.1** – *Let $m\geq 1$ and $z_m=(m+1)\min\{n\in\mathbb{Z}_{\geqslant 0}:\ n\geqslant |z|\}$.*

(a) *For $\sigma\geqslant 3/4$ and $r\geqslant 1/3$, we have*

$$\prod_{p\mid N}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\sum_{\substack{v\geqslant 0\\ u\equiv mv\,(\mathrm{mod}\ 2)}}\frac{\tau_{|z|}(p^v)}{p^{(\sigma+m/2)v}}\leqslant \mathrm{e}^{c[|z|+S_r(N)]}$$

where

$$S_r(N)=\begin{cases}1 & \text{if } r>1/2\\ \log_3(N) & \text{if } r=1/2\\ (\log N)^{1-2r}/\log_2 N & \text{if } r<1/2\end{cases}$$

and the constant $c>0$ does not depend on σ.

(b) *For $\sigma>1$ and $r\geqslant 1/3$ we have*

$$\prod_{p\nmid N}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\sum_{v\geqslant 0}\frac{|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|}{p^{\sigma v}}\leqslant \exp\left(c_\sigma(z_m+3)\right),$$

where $c_\sigma>0$ is a constant depending on σ.

(c) *For $\sigma\in[3/4,1]$ and $r\in[1/3,1]$ we have*

$$\prod_{p\nmid N}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\sum_{v\geqslant 0}\frac{|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|}{p^{\sigma v}}$$
$$\leqslant \exp\left(c(z_m+3)\left[\frac{(z_m+3)^{-1+1/\sigma}-1}{(1-\sigma)\log(z_m+3)}+\log_2(z_m+3)\right]\right)$$

where $c>0$ is a constant not depending on σ.

Proof. (a) Let

$$A_m(p) = \sum_{u\geqslant 0} \frac{\tau(p^u)}{p^{ur}} \sum_{\substack{v\geqslant 0\\ u\equiv mv\,(\mathrm{mod}\ 2)}} \frac{\tau_{|z|}(p^v)}{p^{(\sigma+m/2)v}}.$$

If m is even then by (2.1),

$$A_m(p) = \sum_{u\text{ even}}\sum_{v} = \left(1+\frac{1}{p^{2r}}\right)\left(1-\frac{1}{p^{2r}}\right)^{-2}\left(1-\frac{1}{p^{\sigma+m/2}}\right)^{-|z|}. \tag{13}$$

If m is odd, then we get

$$A_m(p) = \sum_{u\text{ even}}\sum_{v\text{ even}} + \sum_{u\text{ odd}}\sum_{v\text{ odd}} \leqslant \sum_{u\text{ even}}\sum_{v\text{ even}} + \sum_{u\text{ even}}\sum_{v\text{ odd}} \leqslant \sum_{u\text{ even}}\sum_{v}.$$

In both cases, we are led to the bound in the right side of (13). Since $\sigma + m/2 \geq 5/4$ and $r \geq 1/3$, this yields

$$\prod_{p|N} A_m(p) \ll \exp\left(c\left(|z| + \sum_{p|N}\frac{1}{p^{2r}}\right)\right) \leqslant \exp\left(c[|z| + S_r(N)]\right).$$

(b) The proof is similar to [RW07, Page 728]. We separate the product into two parts according to $p^\sigma \leqslant z_m + 3$ or $p^\sigma > z_m + 3$. Using (9) and (10), we have

$$\prod_{p^\sigma > z_m+3} \leqslant \exp\left(\sum_{p^\sigma > z_m+3}\left(\frac{z_m}{p^{\sigma+rm}} + \sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|\right)\right) \tag{14}$$

and

$$\sum_{v\geqslant 2} \leqslant \sum_{u\geqslant 0}\frac{u+1}{p^{ru}}\sum_{v\geqslant 2}\binom{z_m+v-1}{v}\frac{1}{p^{\sigma v}}$$

with

$$\sum_{v\geqslant 2}\binom{z_m+v-1}{v}\frac{1}{p^{\sigma v}} \leqslant \frac{z_m(z_m+1)}{p^{2\sigma}}\sum_{v\geqslant 2}\binom{z_m+v-1}{v-2}\frac{1}{p^{\sigma(v-2)}}$$

so that

$$\sum_{v\geqslant 2} \leqslant \left(1-\frac{1}{p^r}\right)^{-2}\left(\frac{z_m+1}{p^\sigma}\right)^2\left(1-\frac{1}{p^\sigma}\right)^{-z_m-2} \leqslant 4\left(1-\frac{1}{2^{1/3}}\right)^{-2}\left(\frac{z_m+1}{p^\sigma}\right)^2 \tag{15}$$

since $p^\sigma > z_m + 3$. Reporting (15) in (14) leads to

$$\prod_{p^\sigma > z_m+3} \leqslant \exp\left(c(z_m+3)^{1/\sigma}\right).$$

Now we deal with $p^\sigma < z_m + 3$. Using (8), (9) and (10), we have

$$\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\sum_{v\geqslant 0}\frac{|\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}|}{p^{\sigma v}} \leqslant 1 + \frac{z_m}{p^\sigma} + \sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\sum_{u\geqslant 0}\frac{\tau(p^u)}{p^{ru}}\binom{z_m+v-1}{v}.$$

The right hand side, denoted by R, satisfies

$$\begin{aligned}
R &= 1+\frac{z_m}{p^\sigma}+\sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\binom{z_m+v-1}{v}+\sum_{v\geqslant 2}\frac{1}{p^{\sigma v}}\sum_{u\geqslant 1}\frac{\tau(p^u)}{p^{ru}}\binom{z_m+v-1}{v}\\
&=\left(1-\frac{1}{p^\sigma}\right)^{-z_m}+\frac{1}{p^{\sigma+r}}\sum_{u\geqslant 0}\frac{u+2}{p^{ru}}\sum_{v\geqslant 1}\binom{z_m+v}{v+1}\frac{1}{p^{\sigma v}}\\
&\leqslant\left(1-\frac{1}{p^\sigma}\right)^{-z_m}+\frac{2z_m}{p^{\sigma+r}}\left(1-\frac{1}{p^r}\right)^{-2}\sum_{v\geqslant 1}\binom{z_m+v}{v}\frac{1}{p^{\sigma v}}\\
&\leqslant\left(1-\frac{1}{p^\sigma}\right)^{-z_m}+\frac{2z_m}{p^{\sigma+r}}\left(1-\frac{1}{p^r}\right)^{-2}\left(1-\frac{1}{p^\sigma}\right)^{-z_m-1}\\
&\leqslant\left(1-\frac{1}{p^\sigma}\right)^{-z_m-1}\left(1+c\frac{z_m}{p^{\sigma+r}}\right) \qquad (16)
\end{aligned}$$

for some absolute constant $c>0$. Since σ and $\sigma+r$ are greater than 1 it follows that

$$\prod_{p^\sigma<z_m+3}\leqslant\exp\left(c_\sigma(z_m+1)\right).$$

(c) As for establishing (15) we have an absolute constant c such that

$$\begin{aligned}
\prod_{p^\sigma>z_m+3} &\leqslant\exp\left(\sum_{p^\sigma>z_m+3}\frac{z_m}{p^{\sigma+rm}}+c\frac{(z_m+1)^2}{p^{2\sigma}}\right)\\
&\leqslant\exp\left(c\frac{(z_m+3)^{1/\sigma}}{\log(z_m+3)}\right). \qquad (17)
\end{aligned}$$

From (16) we have

$$\prod_{p^\sigma<z_m+3}\leqslant\exp\left(c(z_m+1)\sum_{p^\sigma<z_m+3}\frac{1}{p^\sigma}+\frac{1}{p^{\sigma+r}}\right)$$

and using

$$\sum_{p\leqslant y}\frac{1}{p^\sigma}\ll\log_2 y+\frac{y^{1-\sigma}-1}{(1-\sigma)\log y}$$

valid uniformly for $1/2\leqslant\sigma\leqslant 1$ and $y\geqslant \mathrm{e}^2$ [TW03, Lemma 3.2] we get

$$\prod_{p^\sigma<z_m+3}\leqslant\exp\left(c(z_m+3)\left[\frac{(z_m+3)^{(1-\sigma)/\sigma}-1}{(1-\sigma)\log(z_m+3)}+\log_2(z_m+3)\right]\right). \qquad (18)$$

The result is a consequence of (17) and (18). □

3. Evaluation of the moments

3.1. **Moments in the all level case.** We fix G any function which is holomorphic and bounded in some sufficiently wide vertical strip $|\Re s|\ll 1$, even and normalized by $G(0)=1$. (Note $G'(0)=0$.)

Let $z\in\mathbb{C}$ and $x\geqslant 1$. Define

$$\omega^z_{\mathrm{Sym}^m f}(x)=\sum_{n=1}^{+\infty}\frac{\lambda^z_{\mathrm{Sym}^m f}(n)}{n}\mathrm{e}^{-n/x} \qquad (19)$$

for all $f\in\mathrm{H}^*_k(N)$. We prove the following lemma.

***Lemma* 3.1** – *For all x, z and N we have*

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 \omega^z_{\mathrm{Sym}^m f}(x) = 2\sum_{q\geqslant 1}\frac{\tau(q)}{\sqrt{q}} V_N\left(\frac{q}{N}\right)\sum_{n\geqslant 1}\frac{\mathrm{e}^{-n/x}}{n}\tau_z(n_N)$$
$$\times\left(\prod_{p\mid n^{(N)}}\mu^{z,\nu_p(n)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(q)}}\right)\delta\left(q^{(N)}\mid n^{(N)m}\right)\frac{\square(n_N^m q_N)}{\sqrt{n_N^m q_N}} + O(\mathrm{Err})$$

where

$$V_N(y) = \frac{1}{2i\pi}\int_{(2)}\zeta^{(N)}(1+2w)\left(\frac{\gamma_\infty(1/2+w)}{\gamma_\infty(1/2)}\right)^2\frac{G(w)}{w}y^{-w}\,\mathrm{d}w \tag{20}$$

and

$$\mathrm{Err} = \frac{\tau(N)^2\log N\log_2 N}{N^{1/4}}\, x^{m/4}\left(\log x\right)^{z_m+1}(z_m+m+1)!.$$

Proof. Let $L(s, f\boxplus f) = L(s,f)^2$. This is an L-function in the sense of [IK04, §5.1]. In particular the gamma factor is $\gamma_\infty(s)^2$, the sign of the functional equation is 1, the conductor is N^2 and the n-th Dirichlet coefficient is

$$\lambda_{f\boxplus f}(n) = \sum_{\substack{(q,r)\in\mathbb{Z}^2_{\geqslant 0}\\ qr^2=n}}\mathbb{1}^{(N)}(r)\lambda_f(q)\tau(q).$$

Therefore we can apply [IK04, Theorem 5.3] to obtain

$$L\left(\frac{1}{2}, f\right)^2 = 2\sum_{q\geqslant 1}\frac{\lambda_f(q)\tau(q)}{\sqrt{q}}V_N\left(\frac{q}{N}\right) \tag{21}$$

where

$$V_N(y) = \sum_r\frac{\mathbb{1}^{(N)}(r)}{r}\int_{(3)}(yr^2)^{-u}G(u)\left(\frac{\gamma_\infty(1/2+u)}{\gamma_\infty(1/2)}\right)^2\frac{\mathrm{d}u}{u}$$
$$= \int_{(3)}y^{-u}\zeta^{(N)}(1+2u)G(u)\left(\frac{\gamma_\infty(1/2+u)}{\gamma_\infty(1/2)}\right)^2\frac{\mathrm{d}u}{u}.$$

We have to evaluate

$$T = \sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}}\lambda_f(q)\lambda^z_{\mathrm{Sym}^m f}(n).$$

Similarly to [RW07, Lemma 12] we have

$$T = \frac{\tau_z(n_N)}{\sqrt{n_N^m q_N}}\square(n_N^m q_N)\delta\left(q^{(N)}\mid n^{(N)m}\right)\prod_{p\mid q^{(N)}}\mu^{z,\nu_p(n)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(q)}} \tag{22}$$
$$+ O\left(\frac{\tau(N)^2\log_2 N}{N}n^{m/4}q^{1/4}\tau(q)\log(Nnq)\tau_{(m+1)|z|}(n)\right).$$

From (19), (21) and (22) we deduce

$$\sum_{f\in\mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2\omega^z_{\mathrm{Sym}^m f}(x) = P + E$$

where P is the announced principal term and

$$E = \frac{\tau(N)^2\log_2 N}{N}\sum_q\frac{\tau(q)^2}{q^{1/4}}\log(Nq)V_N\left(\frac{q}{N}\right)\sum_n\frac{\tau_{(m+1)|z|}(n)\log n}{n^{1-m/4}}\mathrm{e}^{-n/x}. \tag{23}$$

We proved in [RW07, Proof of Lemma 16] that the summation over n is

$$\sum_n \ll x^{m/4}(\log x)^{z_m+1}(z_m+m+1)!. \tag{24}$$

Moreover, by (20) and since

$$\sum_q \frac{\tau(q)^2\log(Nq)}{q^s} = \left[\log(N) - \frac{\mathrm{d}}{\mathrm{d}s}\right]\frac{\zeta^4(s)}{\zeta(2s)}$$

we get, after having moved the integration line in V_N from (2) to (7/10) and crossed a pole at $w=3/4$ the majoration

$$\sum_q \frac{\tau(q)^2\log(Nq)}{q^{1/4}} V_N\left(\frac{q}{N}\right) \ll N^{3/4}\log N. \tag{25}$$

The announced error term is a consequence of (23) with (24) and (25). □

We study the principal term exhibited in Lemma 3.1 in the following lemma.

***Lemma* 3.2**– *For any squarefree integer N, any $z\in\mathbb{C}$ and any $x\in\mathbb{R}$ such that*

$$\frac{1}{100m}\log N \leqslant \log x \leqslant \frac{1}{12}\log N$$

we have

$$\begin{aligned}
&\sum_{q\geqslant 1}\frac{\tau(q)}{\sqrt{q}}V_N\left(\frac{q}{N}\right)\sum_{n\geqslant 1}\frac{\mathrm{e}^{-n/x}}{n}\tau_z(n_N)\left(\prod_{p\mid n^{(N)}}\mu^{z,\nu_p(n)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(q)}}\right)\delta\left(q^{(N)}\mid n^{(N)m}\right)\frac{\square(n_N^m q_N)}{\sqrt{n_N^m q_N}}\\
&=\frac{\varphi(N)}{N}A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)\left(\frac{1}{2}\log N+\gamma+\frac{\gamma_\infty'}{\gamma_\infty}\left(\frac{1}{2}\right)+\sum_{p\mid N}\frac{\log p}{p-1}\right)\\
&\quad+\frac{1}{2}B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m;N\right)+O(\mathrm{Err})
\end{aligned}$$

where

$$\mathrm{Err} = \exp\left(c\left[\log_2 N - \frac{\log N}{\log(z_m+3)} + (z_m+3)\log(z_m+3)\right]\right).$$

Proof. We write Σ for the sum to be evaluated:

$$\Sigma = \frac{1}{(2i\pi)^2}\int_{(1)}\int_{(1)} N^w\zeta^{(N)}(1+2w)\left(\frac{\gamma_\infty(1/2+w)}{\gamma_\infty(1/2)}\right)^2 H_N^z(w,s)G(w)\frac{\mathrm{d}w}{w}\Gamma(s)x^s\,\mathrm{d}s \tag{26}$$

with

$$H_N^z(w,s) = \sum_q \frac{\tau(q)}{q^{w+1/2}q_N^{1/2}}\sum_n \frac{\tau_z(n_N)}{n^{s+1}n_N^{m/2}}\delta\left(q^{(N)}\mid n^{(N)m}\right)\square(n_N^m q_N)\prod_{p\mid q^{(N)}}\mu^{z,\nu_p(n)}_{\mathrm{Sym}^m,\mathrm{Sym}^{\nu_p(q)}}.$$

Writing $a = n^{(N)}$, $b = n_N$, $c = q^{(N)}$ and $d = q_N$ we have $H_N^z(w,s) = AB$ where

$$\begin{aligned}A &= \sum_{b\mid N^\infty} \frac{\tau_z(b)}{b^{1+m/2+s}} \sum_{d\mid N^\infty} \frac{\tau(d)}{d^{w+1}} \square(db^m)\\ &= \prod_{p\mid N} \sum_{u\geqslant 0} \frac{\tau(p^u)}{p^{u(w+1)}} \sum_{\substack{v\geqslant 0\\ u\equiv mv \,(\mathrm{mod}\ 2)}} \frac{\tau_z(p^v)}{p^{(s+1+m/2)v}}\\ &= C^z\left(w,s;\frac{1}{p}\right)\end{aligned}$$

by (11) and

$$\begin{aligned}B &= \sum_{(a,N)=1} \frac{1}{a^{s+1}} \sum_{c\mid a^m} \frac{\tau(c)}{c^{w+1/2}} \prod_{p\mid c} \mu^{z,v_p(a)}_{\mathrm{Sym}^m,\mathrm{Sym}^{v_p(c)}}\\ &= \prod_{p\nmid N} \sum_{u\geqslant 0} \frac{\tau(p^u)}{p^{(1/2+w)u}} \sum_{v\geqslant 0} \frac{\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}}{p^{(1+s)v}}\\ &= F^z\left(w,s;\frac{1}{p}\right)\end{aligned}$$

by (12). (Recall that $\mu^{z,v}_{\mathrm{Sym}^m,\mathrm{Sym}^u}$ vanishes when $u > mv$.)

In (26) we shift the w-contour to $\Re w = -1/6$ encountering a simple pole at 0 and obtain

$$\Sigma = P + \frac{1}{2i\pi}\int_{(1)} \Sigma^-(s)\Gamma(s)x^s\,\mathrm{d}s \tag{27}$$

with

$$P = \frac{\varphi(N)}{N}\frac{1}{2i\pi}\int_{(1)}\left[\left(\frac{1}{2}\log N + \gamma + \sum_{p\mid N}\frac{\log p}{p-1} + \frac{\gamma'_\infty(1/2)}{\gamma_\infty(1/2)}\right)H_N^z(0,s) + \frac{1}{2}\frac{\partial}{\partial w}_{\mid(0,s)} H_N^z(w,s)\right]\Gamma(s)x^s\,\mathrm{d}s.$$

We bound $|\Sigma^-|$ as follows. We use lemma 2.1 choosing $\sigma = 2$ and $r = 5/6$ in (a), $r = 1/3$ in (b) to get

$$|\Sigma^-(s)| \ll N^{-1/6}\exp\left[c\left(\frac{(\log N)^{1/3}}{\log_2 N} + z_m\right)\right]$$

hence

$$\Sigma = P + O\left\{xN^{-1/6}\exp\left[c\left(\frac{(\log N)^{1/3}}{\log_2 N} + z_m\right)\right]\right\}.$$

We now treat the integral in the defining expression for P. For this, we replace the segment $[1 - i\log^2 x, 1 + i\log^2 x]$ by the union of the three segments $[1 - i\log^2 x, -\sigma - i\log^2 x]$, $[-\sigma - i\log^2 x, -\sigma + i\log^2 x]$, $[-\sigma + i\log^2 x, 1 + i\log^2 x]$ with $\sigma = 1/\log(|z|+3)$. We shall show that the residue Res of the pole of Γ at 0 provides the main contribution whereas the integral on the new contour enters the error term.

We write

$$P - \mathrm{Res} = A_0 + A_1 + A_2 + B_0 + B_1 + B_2 \tag{28}$$

where

$$\begin{aligned}
\mathrm{Res} &= \frac{\varphi(N)}{N}\left(\frac{\log N}{2}+\gamma+\sum_{p\mid N}\frac{\log p}{p-1}+\frac{\gamma_\infty'}{\gamma_\infty}\left(\frac{1}{2}\right)\right)H_N^z(0,0)+\frac{\varphi(N)}{2N}\frac{\partial}{\partial w}_{|(0,0)}H_N^z(w,s),\\
A_0 &= \frac{\varphi(N)}{N}\left(\frac{\log N}{2}+\gamma+\sum_{p\mid N}\frac{\log p}{p-1}+\frac{\gamma_\infty'}{\gamma_\infty}\left(\frac{1}{2}\right)\right)\frac{1}{2i\pi}\int_{1\pm i\log^2 x}^{1\pm i\infty}H_N^z(0,s)\Gamma(s)x^s\,\mathrm{d}s,\\
B_0 &= \frac{\varphi(N)}{2N}\frac{1}{2i\pi}\int_{1\pm i\log^2 x}^{1\pm i\infty}\frac{\partial}{\partial w}_{|(0,s)}H_N^z(w,s)\Gamma(s)x^s\,\mathrm{d}s,
\end{aligned}$$

and A_1 (resp. B_1) has the same integrand as A_0 (resp. B_0) but the contour is $[1-i\log^2 x, -\sigma-i\log^2 x]$ and A_2 (resp. B_2) has the same integrand as A_0 (resp. B_0) but the contour is $[-\sigma-i\log^2 x, -\sigma+i\log^2 x]$.

From lemma 2.1 (a) and (b) and Stirling formula [IK04, (5.113)] we have

$$A_0 \ll \frac{\varphi(N)\log N}{N}\mathrm{e}^{-\log^2 x+c(z_m+3)}. \tag{29}$$

From lemma 2.1 (a) and (c) and Stirling formula we have

$$A_1 \ll \frac{\varphi(N)\log N}{N}\mathrm{e}^{-\log^2 x+c(z_m+3)\log_2(z_m+3)}. \tag{30}$$

and

$$A_2 \ll \frac{\varphi(N)\log N}{N}\exp\left(-\frac{\log x}{\log(z_m+3)}\right)\mathrm{e}^{c(z_m+3)\log_2(z_m+3)}. \tag{31}$$

The contribution of B_0, B_1 and B_2 are easily seen to be dominated by the ones of A_0, A_1 and A_2 thanks to Cauchy integral formula. Reporting (29), (30) and (31) in (28) and the result in (27) we obtain that Σ is the announced principal term (the residue Res) up to an error term

$$\ll \exp\left(c\left(-\frac{\log N}{\log(z_m+3)}+(z_m+3)\log(z_m+3)+\log_2 N\right)\right).$$

This completes the proof. $\square$

We have now the ingredients to prove theorem A. As in [RW07, pages 743] we have

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^2 L(1,\mathrm{Sym}^m f)^z = \sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^2 \omega_{\mathrm{Sym}^m f}^z(x)+O(\mathrm{Err}) \tag{32}$$

where

$$\begin{aligned}
\mathrm{Err} &= x^{-1/\log_2 N}\mathrm{e}^{D|z|\log_3 N}\log^4 N+\mathrm{e}^{D|z|\log_2 N-\frac{1}{2}\log^2 N}+N^{-1/4}\log^{D|z|}N\\
&\ll \exp\left(D|z|\log_2 N-\alpha\frac{\log N}{\log_2 N}\right)
\end{aligned}$$

by setting $x=N^\alpha$. We have also used

$$\sum_{f\in \mathrm{H}_k^*(N)}^{\mathrm{h}} L\left(\frac{1}{2},f\right)^2 \ll \log N$$

which follows from (21) and Petersson trace formula [ILS00, Corollary 2.10] or [RW07, Lemma 10].

Reporting lemma 3.2, 3.1 in (32) and assuming

$$|z| \leqslant \varepsilon \frac{\log N}{\log_2 N \log_3 N}$$

for $\varepsilon > 0$ small enough (regarding to α) we obtain the theorem.

3.2. **Moments for levels without small prime factors.** Corollary B is a consequence of the following lemma.

***Lemma* 3.3**– *We have*

$$\frac{\varphi(N)}{N} A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m, N\right) = A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)[1 + o_m(1)]$$

and

$$\begin{aligned}\frac{\varphi(N)}{N} B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m, N\right) = B^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)&[1 + o_m(1)] \\ + A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)&o_m(1)\end{aligned}$$

uniformly for

$$\begin{cases} N \in \mathcal{N}\left(\log^2\right) \\ |z| \ll_m \dfrac{\log N}{\log_2 N \log_3 N}. \end{cases} \tag{33}$$

Proof. To prove the first equality, we write

$$\frac{\varphi(N)}{N} A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m, N\right) = A^{2,z}\left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m\right)\frac{E_1(N)}{E_2(N)} \tag{34}$$

with

$$\begin{aligned} E_1(N) &= \prod_{p \mid N} C^z\left(0, 0; \frac{1}{p}\right), \\ E_2(N) &= \prod_{p \mid N} \int_{\mathrm{SU}(2)} D(p^{-1/2}, \mathrm{St}, g)^2 D(p^{-1}, \mathrm{Sym}^m, g)^z \,\mathrm{d}g. \end{aligned}$$

First, we deal with $E_1(N)$. For m even we have

$$\begin{aligned} E_1(N) &= \left(1 + O\left(\frac{\omega(N)}{P^-(N)^2}\right)\right)\left(1 + O\left(\frac{(|z|+1)\omega(N)}{P^-(N)^{1+m/2}}\right)\right) \\ &= 1 + O\left(\frac{(|z|+1)\omega(N)}{P^-(N)^{\min(2, 1+m/2)}}\right) \end{aligned}$$

as soon as the function inside the error term is bounded. If m is odd then

$$\begin{aligned} C^z\left(0, 0; \frac{1}{p}\right) &= \frac{1}{2}\left(1 + \frac{2}{p} + O\left(\frac{1}{p^2}\right)\right)\left(1 + \frac{z}{p^{1+m/2}} + O\left(\frac{(|z|+1)^2}{p^{2+m}}\right)\right) \\ &\quad + \frac{1}{2}\left(1 - \frac{2}{p} + O\left(\frac{1}{p^2}\right)\right)\left(1 - \frac{z}{p^{1+m/2}} + O\left(\frac{(|z|+1)^2}{p^{2+m}}\right)\right) \\ &= 1 + O\left(\frac{(|z|+1)^2}{p^{2+m/2}}\right) \end{aligned}$$

so that

$$E_1(N) = 1 + O\left(\frac{(|z|+1)^2\omega(N)}{P^-(N)^{2+m/2}}\right). \tag{35}$$

From (35) we deduce that

$$E_1(N)=1+o_m(1) \tag{36}$$

if N and z satisfy (33).

To study $E_2(N)$ we define

$$\begin{aligned} e(z,p) &= \int_{\mathrm{SU}(2)} D(p^{-1/2},\mathrm{St},g)^2 D(p^{-1},\mathrm{Sym}^m,g)^z\,\mathrm{d}g \\ &= \sum_{\nu_1=0}^{+\infty} p^{-\nu_1}\sum_{\nu_2=0}^{+\infty} p^{-\nu_2/2}\sum_{u=0}^{\min(m\nu_1,\nu_2)} \mu^{z,\nu_1}_{\mathrm{Sym}^m,\mathrm{Sym}^u}\mu^{2,\nu_2}_{\mathrm{St},\mathrm{Sym}^u} \end{aligned} \tag{37}$$

by orthogonality. Using (8) and (9) we compute the contribution of $\nu_1=1$ and $\nu_2=2$ to (37) and with (10) we obtain

$$\begin{aligned} |e(z,p)-1| &\leqslant \sum_{\nu_2=2}^{+\infty}\binom{3+\nu_2}{\nu_2}\frac{1}{p^{\nu_2/2}}+\frac{|z|}{p}\sum_{\nu_2=m}^{+\infty}\binom{3+\nu_2}{\nu_2}\frac{1}{p^{\nu_2/2}} \\ &\quad+\sum_{\nu_1=2}^{+\infty}\binom{(m+1)|z|+\nu_1-1}{\nu_1}\frac{1}{p^{\nu_1}}\sum_{\nu_2=0}^{+\infty}\binom{3+\nu_2}{\nu_2}\frac{1}{p^{\nu_2/2}} \\ &\ll_m \frac{1}{p}+\frac{|z|}{p^{1+m/2}}+\frac{|z|(|z|+1)}{p^2}. \end{aligned}$$

It follows that

$$E_2(N)=1+O\left(\frac{\omega(N)}{P^-(N)}\left(1+\frac{|z|}{P^-(N)^{m/2}}+\frac{(|z|+1)^2}{P^-(N)}\right)\right)=1+o_m(1) \tag{38}$$

if N and z satisfy (33). The first result of the lemma follows from (34), (36) and (38).

We consider now $B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m,N\right)$. We begin in considering

$$F_N^z(w,0)=\left(\prod_{\substack{p\in\mathscr{P}\\ p\nmid N}} F^z\left(w,0;\frac{1}{p}\right)\prod_{\substack{p\in\mathscr{P}\\ p\mid N}} C^z\left(w,0;\frac{1}{p}\right)\right)$$

with enough uniformity in some fixed neighbourhood of w to be authorized to apply Cauchy integral formula. We write $F_N^z(w,0)=F_1^z(w,0)Q_N(w)$ with

$$Q_N(w)=Q_N^{(1)}(w)/Q_N^{(2)}(w)$$

and

$$Q_N^{(1)}(w)=\prod_{p\mid N} C^z\left(w,0,\frac{1}{p}\right),\qquad Q_N^{(2)}(w)=\prod_{p\mid N} F^z\left(w,0,\frac{1}{p}\right).$$

As for $E_1(N)$ and $E_2(N)$ we compute

$$Q_N^{(1)}(w)=1+O_\varepsilon\left(\frac{\omega(N)}{P^-(N)^{1-\varepsilon}}\left(1+\frac{|z|}{P^-(N)^{m/2+\varepsilon}}\right)\right) \tag{39}$$

and

$$\frac{N}{\varphi(N)}Q_N^{(2)}(w)=1+O_\varepsilon\left(\frac{\omega(N)}{P^-(N)^{1-2\varepsilon}}\left(1+\frac{|z|}{P^-(N)^{1/2+\varepsilon}}+\frac{(|z|+1)^2}{P^-(N)^{1+2\varepsilon}}\right)\right) \tag{40}$$

the constant implied by the error term being independant of w such that $\Re w>-\varepsilon$. It follows in particular that

$$Q_N(0)=1+o_m(1) \tag{41}$$

if N and z satisfy (33). Denote $C(0,\varepsilon)$ the circle of centre 0 and radius ε. We have

$$\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} F_N^z(w,0) = \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} F_1^z(w,0)Q_N(0) + F_1^z(0,0)\cdot\frac{1}{2i\pi}\int_{C(0,\varepsilon)} Q_N(w)\frac{\mathrm{d}w}{w^2} \tag{42}$$

and from the uniformity in w in (39) and (40) we deduce

$$\frac{1}{2i\pi}\int_{C(0,\varepsilon)} Q_N(w)\frac{\mathrm{d}w}{w^2} = o(1). \tag{43}$$

Reporting (41) and (43) in (42) we obtain the second result of the lemma. □

4. Behavior for the asymptotic real moments

4.1. **Behavior of the main term.** The aim of this section is to prove Theorem C. In fact we shall establish a more general result (see Proposition 4.1 below). Write

$$D_m(\theta,t) := D(t,\mathrm{Sym}^m,g) = \prod_{j=0}^{m}\left(1-\mathrm{e}^{\mathrm{i}(m-2j)\theta}t\right)^{-1}, \tag{44}$$

and

$$F_m^{\ell,z}(w,s;t) := \left(1-t^{1+2w}\right)^{\frac{\ell(\ell-1)}{2}}\frac{2}{\pi}\int_0^\pi D_1\left(\theta,t^{1/2+w}\right)^\ell D_m\left(\theta,t^{1+s}\right)^z \sin^2\theta\,\mathrm{d}\theta.$$

so that

$$F^z(w,s;t) = F_m^{2,z}(w,s;t).$$

***Proposition* 4.1–** *Let $J\geqslant 1$, $\ell\geqslant 0$ and $m\geqslant 1$ be three fixed integers. Then we have*

$$\sum_{p\leqslant y}\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = z\left\{(m+1)\log_2 z + (m+1)\gamma + \sum_{j=1}^{J}\frac{a_j}{(\log z)^j} + O\left(\frac{1}{(\log z)^{J+1}}\right)\right\}$$

uniformly for $y\geqslant z^{3/2}\geqslant 10$, where γ is the Euler constant and a_j is defined as in Theorem C.

Since

$$A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^m\right) = \prod_{p\in\mathscr{P}} F_m^{2,z}\left(0,0;\frac{1}{p}\right),$$

Theorem C is an immediate consequence of Proposition 4.1 by taking $\ell=2$ and making $y\to+\infty$.

In order to prove this proposition, we first establish some preliminary lemmas.

***Lemma* 4.2–** *Let $g_m(t)$ and $\widetilde{g}_m(t)$ be defined as in* (2) *and* (3). *Then*

$$\widetilde{g}_m(t) \ll \begin{cases} t^2 & \text{if } 0\leqslant t<1,\\ \log(2t) & \text{if } t\geqslant 1,\end{cases}$$

and

$$\widetilde{g}'_m(t) \ll \begin{cases} t & \text{if } 0\leqslant t<1,\\ t^{-1} & \text{if } t\geqslant 1.\end{cases}$$

Proof. When $t \geqslant 0$, we can write

$$e^{tX_m(2\cos\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t\sin((m+1)\theta)}{\sin\theta}\right)^n.$$

From this we deduce, for $0 \leqslant t < 1$,

$$\begin{aligned}\widetilde{g}_m(t) &= \log\left(1+\sum_{n=2}^{\infty}\frac{t^n}{n!}\frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin((m+1)\theta)}{\sin\theta}\right)^n \sin^2\theta\,\mathrm{d}\theta\right)\\ &= \log\big(1+t^2+O\big(t^3\big)\big) \asymp t^2\end{aligned}$$

and

$$\widetilde{g}'_m(t) \asymp t.$$

Let C_m be the maximum of $2X'_m(x)$ in $[-2,2]$. Then, since $X_m(2) = m+1$, we have

$$0 \leqslant m+1-X_m(2\cos\theta) \leqslant C_m(1-\cos\theta)$$

for every $\theta \in [0,\pi]$. Thus for $t \geqslant 1$, we have by (3) and (5),

$$\widetilde{g}'_m(t) = -\frac{\displaystyle\int_0^{\pi} e^{tX_m(2\cos\theta)}(m+1-X_m(2\cos\theta))\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta} \ll_m |\widetilde{h}_m(t)|.$$

Now (54) of Lemma 4.6 below implies $\widetilde{g}'_m(t) \ll t^{-1}$ for $t \geqslant 1$. From this we immediately deduce $\widetilde{g}_m(t) \ll \log(2t)$ for $t \geqslant 1$. □

Lemma 4.3– *Let $m \geqslant 1$ be a fixed integer. Then we have*

$$\int_0^{\pi} e^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta \ll t\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta \tag{45}$$

and

$$\frac{2}{\pi}\int_0^{\pi} e^{tX_m(2\cos\theta)}\cos^2\theta\sin^2\theta\,\mathrm{d}\theta = \left\{\frac{1}{4}+O(t)\right\}\frac{2}{\pi}\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta \tag{46}$$

uniformly for $t \geqslant 0$. The implied constants depend on m only.

Proof. First we note that these estimates are trivial for $t \geqslant 1$, so we suppose that $0 \leqslant t \leqslant 1$. In view of the following relations:

$$\frac{2}{\pi}\int_0^{\pi}\cos^n\theta\sin^2\theta\,\mathrm{d}\theta = \begin{cases} 0 & \text{if } n \text{ is odd}\\ 1 & \text{if } n=0\\ 2\dfrac{(2r-1)!!}{(2r+2)!!} & \text{if } n=2r\end{cases}$$

(with $n!! := n\cdot(n-2)\cdots$) and $e^{tX_m(2\cos\theta)} = 1+O(t)$, it follows that

$$\begin{aligned}\int_0^{\pi} e^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta &\ll t\int_0^{\pi}|\cos\theta|\sin^2\theta\,\mathrm{d}\theta\\ &\ll t\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.\end{aligned}$$

Similarly

$$\frac{2}{\pi}\int_0^{\pi} e^{tX_m(2\cos\theta)}\cos^2\theta\sin^2\theta\,\mathrm{d}\theta = \frac{1}{4}+O(t)$$

which implies (46). □

Lemma 4.4– *Let $\ell \geqslant 0$ and $m \geqslant 1$ be two fixed integers. Suppose $z \geq 4$ is real. Then we have*

$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = -(m+1)z\log\left(1-\frac{1}{p}\right) + O\big(\log z\big) \tag{47}$$

uniformly for $2 \leqslant p \leqslant \sqrt{z}$; and

$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = g_m\left(\frac{z}{p}\right) + O\left(\frac{z}{p^{3/2}}\right) \tag{48}$$

uniformly for $p \geqslant \sqrt{z} \geqslant 2$. The implied constants depend on ℓ and m only.

Proof. We have

$$\prod_{j=0}^{m}\left(1-\frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p}\right) = \sum_{\nu=0}^{m+1}\frac{(-1)^\nu}{p^\nu}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\mathrm{e}^{\mathrm{i}(\nu m-2j_1-\cdots-2j_\nu)\theta}.$$

Since the left-hand side is real and

$$\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\mathrm{e}^{\mathrm{i}(\nu m-2j_1-\cdots-2j_\nu)\theta} = 1 \qquad (\nu = 0, m+1),$$

it follows that, with notation $\mathbf{j}_\nu = (j_1,\dots,j_\nu)$ and $\ell_{\mathbf{j}_\nu}^m = \nu m - 2j_1 - \cdots - 2j_\nu$,

$$\begin{aligned}
\prod_{j=0}^{m}\left(1-\frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p}\right) &= \sum_{\nu=0}^{m+1}\frac{(-1)^\nu}{p^\nu}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\cos\left(\ell_{\mathbf{j}_\nu}^m\theta\right)\\
&= \left(1-\frac{1}{p}\right)^{m+1} + \sum_{\nu=1}^{m}\frac{(-1)^{\nu-1}}{p^\nu}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}\left\{1-\cos\left(\ell_{\mathbf{j}_\nu}^m\theta\right)\right\}\\
&= \left(1-\frac{1}{p}\right)^{m+1} + \sum_{\nu=1}^{m}\frac{(-1)^{\nu-1}}{p^\nu}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}2\sin^2\left(\ell_{\mathbf{j}_\nu}^m\theta/2\right).
\end{aligned}$$

Introducing the notation

$$\widetilde{D}_m\left(\theta,p^{-1}\right) := 1 + \left(1-\frac{1}{p}\right)^{-(m+1)}\sum_{\nu=1}^{m}\frac{(-1)^{\nu-1}}{p^\nu}\sum_{0\leqslant j_1<\cdots<j_\nu\leqslant m}2\sin^2\left(\ell_{\mathbf{j}_\nu}^m\theta/2\right),$$

we can write

$$\prod_{j=0}^{m}\left(1-\frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p}\right) = \left(1-\frac{1}{p}\right)^{m+1}\widetilde{D}_m\left(\theta,p^{-1}\right)$$

and

$$F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = \left(1-\frac{1}{p}\right)^{-(m+1)z+\ell(\ell-1)/2}\check{F}_m^{\ell,z}(p)$$

with

$$\check{F}_m^{\ell,z}(p) := \frac{2}{\pi}\int_0^\pi D_1\left(\theta,p^{-1/2}\right)^\ell\,\widetilde{D}_m\left(\theta,p^{-1}\right)^{-z}\sin^2\theta\,\mathrm{d}\theta.$$

Observing the nonnegativity of the integrand, we infer that for some suitably small positive constant δ,

$$\begin{aligned}\check{F}_m^{\ell,z}(p) &\geqslant \frac{2}{\pi}\left(1-\frac{1}{\sqrt{2}}\right)^{2\ell}\int_0^{\delta\sqrt{p/z}}\left(1+\frac{c_m\theta^2}{p}\right)^{-z}\theta^2\,\mathrm{d}\theta\\ &\gg \int_0^{\delta\sqrt{p/z}}\left(1+\frac{c_m\delta^2}{z}\right)^{-z}\theta^2\,\mathrm{d}\theta\\ &\gg \left(1+\frac{c_m\delta^2}{z}\right)^{-z}\left(\frac{p}{z}\right)^{3/2}\\ &\gg_m \left(\frac{p}{z}\right)^{3/2}\end{aligned}$$

for $p\leqslant z$. On the other hand, it is obvious that

$$\left|\widetilde{D}_m\left(\theta,p^{-1}\right)^{-1}\right|\leqslant 1 \qquad\text{and}\qquad \check{F}_m^{\ell,z}(p)\ll 1$$

uniformly for $p\leqslant\sqrt{z}$. By combining these estimates, we find that

$$\begin{aligned}\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) &= \log\left(1-\frac{1}{p}\right)^{-(m+1)z+\ell(\ell-1)/2}+\log\check{F}_m^{\ell,z}(p)\\ &= (m+1)z\log\left(1-\frac{1}{p}\right)^{-1}+O\big(\log z\big)\end{aligned}$$

for $p\leqslant\sqrt{z}$.

Next we prove (48). In view of (44) and (9), it is easy to see that

$$D_m\left(\theta,p^{-1}\right)^z = \mathrm{e}^{(z/p)X_m(2\cos\theta)}\left\{1+O\left(\frac{z}{p^2}\right)\right\}\qquad\left(p\geqslant\sqrt{z}\right),\tag{49}$$

where the implied constant depends on m at most. Thus for $p\geqslant\sqrt{z}$, we can write

$$F_m^{\ell,z}\left(0,0;\frac{1}{p}\right)=\left\{1+O\left(\frac{z}{p^2}\right)\right\}\left(1-\frac{1}{p}\right)^{\ell(\ell-1)/2}\widetilde{F}_m^{\ell,z}(p)$$

with

$$\widetilde{F}_m^{\ell,z}(p):=\frac{2}{\pi}\int_0^{\pi}D_1\left(\theta,p^{-1/2}\right)^{\ell}\,\mathrm{e}^{(z/p)X_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.$$

Since

$$D_1\left(\theta,p^{-1/2}\right)^{\ell}=1+\frac{2\ell\cos\theta}{p^{1/2}}+\frac{2(\ell+1)\ell\cos^2\theta-\ell}{p}+O\left(\frac{1}{p^{3/2}}\right)\tag{50}$$

where the implied constant depends on ℓ at most, (45) and (46) of Lemma 4.3 allow us to deduce that

$$\widetilde{F}_m^{\ell,z}(p)=\left\{1+\frac{\ell(\ell-1)}{2p}+O\left(\frac{z}{p^{3/2}}\right)\right\}\frac{2}{\pi}\int_0^{\pi}\mathrm{e}^{(z/p)X_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.$$

Inserting it into the preceding relation, we easily obtain (48). □

Now we are ready to prove Proposition 4.1. From (47) and (48), we deduce that for $y\geq z^{3/2}$,

$$\sum_{p\leqslant y}\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right)=(m+1)z\sum_{p\leqslant\sqrt{z}}\log\left(1-\frac{1}{p}\right)^{-1}+\sum_{\sqrt{z}<p\leqslant y}g_m\left(\frac{z}{p}\right)+O\left(\frac{z^{3/4}}{\log z}\right).$$

In view of (2), (3) and the following estimate

$$\sum_{\sqrt{z}<p\leqslant z}\left\{(m+1)z\log\left[\left(1-\frac{1}{p}\right)^{-1}\right]-(m+1)\frac{z}{p}\right\}\ll\frac{\sqrt{z}}{\log z},$$

the last asymptotic formula can be written as

$$\begin{aligned}\text{(51)}\qquad \sum_{p\leqslant y}\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right)\\ =(m+1)z\sum_{p\leqslant z}\log\left(1-\frac{1}{p}\right)^{-1}+\sum_{\sqrt{z}<p\leqslant y}\widetilde{g}_m\left(\frac{z}{p}\right)+O\left(\frac{z^{3/4}}{\log z}\right).\end{aligned}$$

By the prime number theorem, it follows that

$$\text{(52)}\qquad \sum_{\sqrt{z}<p\leqslant y}\widetilde{g}_m\left(\frac{z}{p}\right)=\int_{\sqrt{z}}^{y}\widetilde{g}_m\left(\frac{z}{u}\right)\mathrm{d}\sum_{p\leqslant u}1=\int_{\sqrt{z}}^{y}\frac{\widetilde{g}_m(z/u)}{\log u}\,\mathrm{d}u+R_1,$$

where

$$R_1:=\int_{\sqrt{z}}^{y}\widetilde{g}_m\left(\frac{z}{u}\right)\mathrm{d}O\left(u\mathrm{e}^{-2\sqrt{\log u}}\right).$$

In view of Lemma 4.2, a simple partial integration gives us

$$R_1\ll z\mathrm{e}^{-\sqrt{\log z}}.$$

In order to evaluate the last integral of (52), we use the change of variables $t=z/u$ to write

$$\begin{aligned}\int_{\sqrt{z}}^{y}\frac{\widetilde{g}_m(z/u)}{\log u}\,\mathrm{d}u&=z\int_{z/y}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2\log(z/t)}\,\mathrm{d}t\\&=z\int_{1/\sqrt{z}}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2\log(z/t)}\,\mathrm{d}t+O(R_2)\end{aligned}$$

where

$$R_2:=z\int_{z/y}^{1/\sqrt{z}}\frac{|\widetilde{g}_m(t)|}{t^2\log(z/t)}\,\mathrm{d}t\ll\frac{z^{1/2}}{\log z}$$

by using Lemma 4.2. On the other hand, we have

$$\begin{aligned}\int_{1/\sqrt{z}}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2\log(z/t)}\,\mathrm{d}t&=\frac{1}{\log z}\int_{1/\sqrt{z}}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2(1-(\log t)/\log z)}\,\mathrm{d}t\\&=\sum_{j=1}^{J}\frac{1}{(\log z)^j}\int_{1/\sqrt{z}}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2}(\log t)^{j-1}\,\mathrm{d}t+O_J\left(\frac{1}{(\log z)^{J+1}}\right).\end{aligned}$$

Extending the interval of integration $[1/\sqrt{z},\sqrt{z}]$ to $(0,\infty)$ and bounding the contributions of $(0,1/\sqrt{z}]$ and $[\sqrt{z},\infty)$ by using Lemma 4.2, we have

$$\int_{1/\sqrt{z}}^{\sqrt{z}}\frac{\widetilde{g}_m(t)}{t^2}(\log t)^{j-1}\,\mathrm{d}t=a_j+O\left(\frac{(\log z)^j}{\sqrt{z}}\right).$$

Combining these estimates, we find that

$$\text{(53)}\qquad \sum_{\sqrt{z}<p\leqslant y}\widetilde{g}_m\left(\frac{z}{p}\right)=z\left\{\sum_{j=1}^{J}\frac{a_j}{(\log z)^j}+O_J\left(\frac{1}{(\log z)^{J+1}}\right)\right\}.$$

Now the desired result follows from (51), (53) and the prime number theorem in the form

$$\sum_{p\leqslant z} \log\left(1-\frac{1}{p}\right)^{-1} = \log_2 z + \gamma + O\left(\mathrm{e}^{-2\sqrt{\log z}}\right).$$

This completes the proof.

4.2. **Behavior of the constant term.** The aim of this section is to prove Theorem D. We shall prove a slightly more general result, i.e. Proposition 4.5. Clearly Theorem D is its simple consequence with the choice of $\ell = 2$.

Let $\ell \geqslant 0$ and $m \geqslant 1$ be two fixed integers. Define

$$B_m(w) = B_m(w,z,p) := \frac{2}{\pi}\int_0^\pi D_1\left(\theta, p^{-(1/2+w)}\right)^\ell D_m\left(\theta, p^{-1}\right)^z \sin^2\theta \,\mathrm{d}\theta$$

so that

$$F_m^{\ell,z}\left(w,0;p^{-1}\right) = \left(1-p^{-(1+2w)}\right)^{\ell(\ell-1)/2} B_m(w).$$

***Proposition* 4.5**– *Let $\ell \geqslant 0$. We have*

$$\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right) \ll \log z$$

uniformly for $y \geqslant z \geqslant 10$ if m is even; and

$$\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right) = \sqrt{z}\left\{b_{\ell,m} + O\left(\mathrm{e}^{-\sqrt{\log z}}\right)\right\}$$

uniformly for $y \geqslant z\mathrm{e}^{2\sqrt{\log z}} \geqslant 10$ if m is odd, where

$$b_{\ell,m} := -2\ell\left(2+\int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t\right).$$

The implied constant depends on ℓ and m only.

We need preliminary lemmas.

***Lemma* 4.6**– *Let $h_m(t)$ and $\widetilde{h}_m(t)$ be defined as in* (4) *and* (5). *Then*

$$\widetilde{h}_m(t) \ll \begin{cases} t & \text{if } 0\leqslant t<1,\\ t^{-1} & \text{if } t\geqslant 1,\end{cases} \qquad \widetilde{h}'_m(t) \ll \begin{cases} 1 & \text{if } 0\leqslant t<1,\\ t^{-1} & \text{if } t\geqslant 1.\end{cases} \tag{54}$$

Further if m is even, then

$$h_m(t) = 0 \qquad (t\geqslant 0). \tag{55}$$

Proof. Equation (55) follows from

$$h_m(t) = \int_{-\pi/2}^{\pi/2} \mathrm{e}^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta \qquad (m \text{ even})$$

by parity. The estimates of (54) with $0\leqslant t\leqslant 1$ are equivalent to (45). Next we prove $\widetilde{h}_m(t) \ll t^{-1}$ for $t\geqslant 1$, i.e.

$$\frac{\displaystyle\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\displaystyle\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta} \ll \frac{1}{t}. \tag{56}$$

From the power series expansion, we have

$$X_m(2\cos\theta) = (m+1) - \frac{m(m+1)(m+2)}{6}\theta^2 + O_m\left(\theta^4\right),$$

and hence there exists $\delta = \delta_m \in (0, \pi/(3(m+1)))$ such that for all $0 \leqslant \theta \leqslant \delta$,

$$(m+1) - \frac{(m+2)^3}{6}\theta^2 < X_m(2\cos\theta) < (m+1) - \frac{1}{6}\theta^2. \tag{57}$$

Since $\theta \mapsto X_m(2\cos\theta)$ is continuous on the compact $[\delta, 2]$ where its values are strictly less than $m+1$, there exists $\alpha_m \in (0, m+1)$ such that

$$|X_m(2\cos\theta)| \leqslant \alpha_m \qquad (\delta \leqslant \theta \leqslant \pi/2). \tag{58}$$

We give a lower bound to the denominator of the fraction in (56). As the integrand is nonnegative, we infer from (57) that

$$\begin{aligned} \int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} \sin^2\theta\, d\theta &\geqslant \int_0^\delta \mathrm{e}^{tX_m(2\cos\theta)} \sin^2\theta\, d\theta \\ &\gg \mathrm{e}^{(m+1)t} \int_0^\delta \mathrm{e}^{-c_m t\theta^2}\theta^2\, d\theta \gg_m \frac{\mathrm{e}^{(m+1)t}}{t^{3/2}} \end{aligned} \tag{59}$$

where the implied constant in $\gg_m$ depends on m only. For the numerator in the left-hand side of (56), we write

$$\begin{aligned} \int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta &= \int_0^{\pi/2} \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta \\ &\quad + \int_0^{\pi/2} \mathrm{e}^{-tX_m(2\cos\theta)}(1+\cos\theta)\sin^2\theta\,\mathrm{d}\theta. \end{aligned}$$

Since $X_m(2\cos\theta) \geqslant 0$ for $\theta \in [0, \pi/(2(m+1))]$, we deduce with (58) that

$$\begin{aligned} \int_0^{\pi/2} \mathrm{e}^{-tX_m(2\cos\theta)}(1+\cos\theta)\sin^2\theta\,\mathrm{d}\theta &\ll \int_0^{\pi/(2(m+1))} \mathrm{d}\theta + \int_{\pi/(2(m+1))}^{\pi/2} \mathrm{e}^{t\alpha_m}\,\mathrm{d}\theta \\ &\ll \mathrm{e}^{\alpha_m t}, \end{aligned}$$

which is negligible in comparison with (59). Splitting at $\theta = \delta$ and applying (57) and (58), we have

$$\begin{aligned} \int_0^{\pi/2} \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta &\ll \mathrm{e}^{(m+1)t}\int_0^\delta \mathrm{e}^{-\frac{1}{6}t\theta^2}\theta^4\,\mathrm{d}\theta + \int_\delta^{\pi/2} \mathrm{e}^{\alpha_m t}\,\mathrm{d}\theta \\ &\ll t^{-5/2}\mathrm{e}^{(m+1)t} + \mathrm{e}^{\alpha_m t}. \end{aligned}$$

The desired estimate in (56) follows with (59) and the fact $\alpha_m < m+1$.

A direct differentiation shows that

$$\begin{aligned} \widetilde{h}_m'(t) = &\frac{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} X_m(2\cos\theta)\sin^2\theta\,\mathrm{d}\theta \int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\left(\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta\right)^2} \\ &- \frac{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} X_m(2\cos\theta)(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta}{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta} \qquad (t \geqslant 1). \end{aligned}$$

Using the nonnegativity, we see that

$$\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} X_m(2\cos\theta)\sin^2\theta\,\mathrm{d}\theta \ll \int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta,$$

and

$$\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} X_m(2\cos\theta)(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta \\ \ll \int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta.$$

Therefore (56) implies $\widetilde{h}_m'(t) \ll t^{-1}$ for $t \geqslant 1$. □

Lemma 4.7 – *Let $\ell \geqslant 0$ and $m \geqslant 1$ be two fixed integers. Then we have*

$$\frac{B_m'(0)}{B_m(0)} \ll \frac{\log p}{p^{1/2}} \tag{60}$$

uniformly for all p and $z \geqslant 1$; and

$$\frac{B_m'(0)}{B_m(0)} = -2\ell\frac{\log p}{p^{1/2}}h_m\left(\frac{z}{p}\right) - \ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right) \tag{61}$$

uniformly for $p \geqslant z^{2/3}$. The implied constants depend on ℓ and m only.

Proof. We have

$$B_m'(0) = \\ -2\ell\frac{2}{\pi}\int_0^\pi D_1\left(\theta, p^{-1/2}\right)^{\ell+1}\left(\frac{\cos\theta}{p^{1/2}} - \frac{1}{p}\right)(\log p)D_m\left(\theta, p^{-1}\right)^z\sin^2\theta\,\mathrm{d}\theta \tag{62}$$

hence

$$B_m'(0) \ll \frac{\log p}{p^{1/2}}\int_0^\pi D_m\left(\theta, p^{-1}\right)^z\sin^2\theta\,\mathrm{d}\theta.$$

This implies (60), since

$$B_m(0) = \left\{1 + O\left(\frac{1}{p^{1/2}}\right)\right\}\frac{2}{\pi}\int_0^\pi D_m\left(\theta, p^{-1}\right)^z\sin^2\theta\,\mathrm{d}\theta.$$

In view of (50), it follows that

$$D_1\left(\theta, p^{-1/2}\right)^{\ell+1}\left(\frac{\cos\theta}{p^{1/2}} - \frac{1}{p}\right) = \frac{\cos\theta}{p^{1/2}} + \frac{2(\ell+1)\cos^2\theta - 1}{p} + O\left(\frac{1}{p^{3/2}}\right). \tag{63}$$

By using it, (49) and (46) of Lemma 4.3, we can deduce, for $p \geqslant \sqrt{z}$,

$$B_m'(0) = -2\ell\frac{\log p}{p^{1/2}}\frac{2}{\pi}\int_0^\pi \mathrm{e}^{(z/p)X_m(2\cos\theta)}\cos\theta\sin^2\theta\,\mathrm{d}\theta \\ - \left\{\ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right)\right\}\frac{2}{\pi}\int_0^\pi \mathrm{e}^{(z/p)X_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.$$

Under the same condition, thanks to (49) and (45), we have

$$B_m(0) = \left\{1 + O\left(\frac{1}{p} + \frac{z}{p^{3/2}}\right)\right\}\frac{2}{\pi}\int_0^\pi \mathrm{e}^{(z/p)X_m(2\cos\theta)}\sin^2\theta\,\mathrm{d}\theta.$$

Combining these, we obtain (61). □

Lemma 4.8– *Let $\ell \geqslant 0$ and $m \equiv 0 \pmod 2$ be two fixed integers. Then we have*

$$\frac{B_m'(0)}{B_m(0)} \ll \frac{\log p}{p} \tag{64}$$

uniformly for all p and $z \geqslant 1$, and

$$\frac{B_m'(0)}{B_m(0)} = -\ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right) \tag{65}$$

uniformly for $p \geqslant z^{2/3}$. The implied constants depend on ℓ and m only.

Proof. Equation (64) follows from (62) and (63) since, by parity consideration we have

$$\int_0^{\pi} (\cos\theta) D_m\left(\theta, p^{-1}\right)^z \sin^2\theta \,\mathrm{d}\theta = 0.$$

Equation (65) is an immediate consequence of (61) since $h_m(t) = 0$ when m is even. □

Now we are ready to prove Proposition 4.5. If m is even, we apply Lemma 4.8 to

$$\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w, 0; \frac{1}{p}\right) = \sum_{p\leqslant y} \ell(\ell-1)\frac{\log p}{p-1} + \sum_{p\leqslant y} \frac{B_m'(0)}{B_m(0)}$$

and obtain

$$\begin{aligned}\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w, 0; \frac{1}{p}\right) &= \sum_{p\leqslant z} \ell(\ell-1)\frac{\log p}{p-1} + \sum_{p\leqslant z} \frac{B_m'(0)}{B_m(0)} \\ &\quad + \sum_{z<p\leqslant y}\left\{\ell(\ell-1)\left(\frac{\log p}{p-1} - \frac{\log p}{p}\right) + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right)\right\} \ll \log z.\end{aligned}$$

When m is odd, by using (60) of Lemma 4.7 for $p \leqslant z^{2/3}$ and (61) for $z^{2/3} < p \leqslant y$, we obtain

$$\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w, 0; \frac{1}{p}\right) = -2\ell \sum_{z^{2/3}<p\leqslant y} \frac{\log p}{p^{1/2}} h_m\left(\frac{z}{p}\right) + O\left(z^{1/3}\log z\right)$$

so that

$$\begin{aligned}\sum_{p\leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} \log F_m^{\ell,z}\left(w, 0; \frac{1}{p}\right) &= \\ -2\ell\left\{\sum_{p\leqslant z}\frac{\log p}{p^{1/2}} + \sum_{z^{2/3}<p\leqslant y}\frac{\log p}{p^{1/2}}\widetilde{h}_m\left(\frac{z}{p}\right)\right\} &+ O\left(z^{1/3}\log z\right).\end{aligned} \tag{66}$$

By using the prime number theorem, it follows by integration by parts that

$$\begin{aligned}\sum_{z^{2/3}<p\leqslant y} \frac{\log p}{p^{1/2}}\widetilde{h}_m\left(\frac{z}{p}\right) &= \int_{z^{2/3}}^{y} \frac{\widetilde{h}_m(z/u)}{u^{1/2}}\,\mathrm{d}u + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right) \\ &= \sqrt{z}\int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right)\end{aligned}$$

with the help of Lemma 4.6, provided $y \geqslant z\mathrm{e}^{2\sqrt{\log z}}$. Combining these yields

$$\sum_{z^{2/3}<p\leqslant y} \frac{\log p}{p^{1/2}}\widetilde{h}_m\left(\frac{z}{p}\right) = \sqrt{z}\int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}}\,\mathrm{d}t + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right). \tag{67}$$

Now the required result is a simple consequence of (66) and(67) and the prime number theorem.

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